☒ §16.2-3 The three first order operators of classical physics –

- The equations of classical physics (Fluid Dynamics, Electromagnetism etc.) are formulated in terms of three 1st order operators: Grad, Div, Curl

Defn: An operator is a function whose inputs & outputs are functions

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

(1) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

(2) $\mathrm{Div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \doteq \nabla \cdot \vec{F}$

(3) $\mathrm{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = (P_y - N_z)\underset{\sim}{i} - (P_x - M_z)\underset{\sim}{j} + (N_x - M_y)\underset{\sim}{k}$

$\doteq \nabla \times \vec{F}$

- Each operator ∇, Div, Curl has a 3 dimensional version of the Fundamental Theorem of Calculus (FTC) that goes with it

(1) $\nabla \iff$ "Conservation of Energy"

(2) Div $\iff$ Divergence Theorem

(3) Curl $\iff$ Stokes Theorem

- Recall: FTC says "the integral of a derivative reduces to an undifferentiated function evaluated on the boundary"

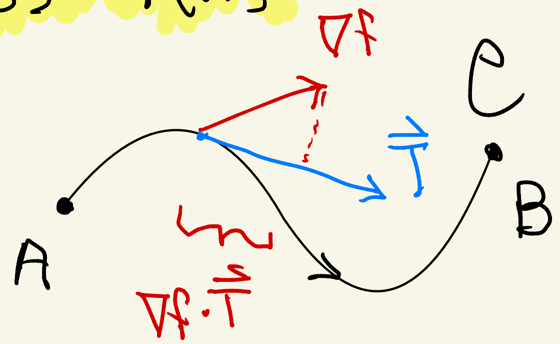
Simplest Case: Math 21B

$$\underbrace{\int_a^b f'(x)\,dx}_{\text{integral of the derivative on } [a,b]} = \underbrace{f(b) - f(a)}$$

undifferentiated function f evaluated on the boundary $x = a, b$

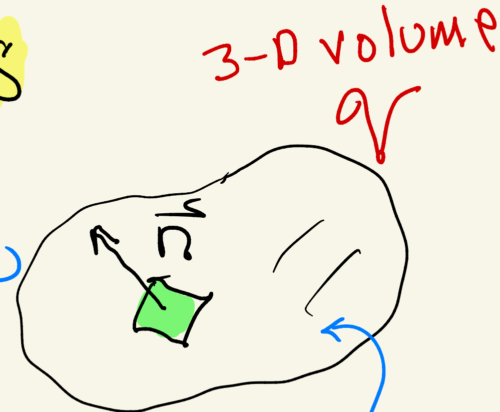
(1) ∇: Conservation of Energy

$$\int_C \nabla f\cdot\vec{T}\,ds = f(B)-f(A)$$



(2) Div: Divergence Thm

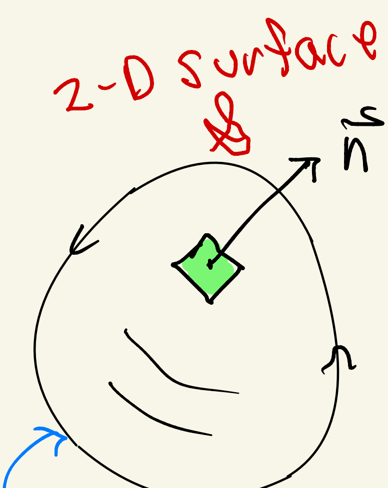
$$\underbrace{\iiint_V \mathrm{Div}\,\vec{F}\,dV}_{\text{3-D triple integral defined in §15}} = \underbrace{\iint_S \vec{F}\cdot\vec{n}\,dS}_{\text{2D Surface Integral (Flux)}}$$



Boundary Surface S

(3) Curl: Stokes Theorem

$$\underbrace{\iint_S \mathrm{Curl}\,\vec{F}\cdot\vec{n}\,dS}_{\text{2-D surface integral (Flux)}} = \underbrace{\int_C \vec{F}\cdot\vec{T}\,ds}_{\text{line integral around closed boundary}}$$



1-D boundary Curve C

III. Our First Generalization of FTC

(I) The FTC associated with the Gradient:

$$\nabla f = \overrightarrow{\underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}_{\text{output } \nabla f : \mathbb{R}^3 \to \mathbb{R}^3}}$$

"input" $(f : \mathbb{R}^3 \to \mathbb{R})$

We use the notation

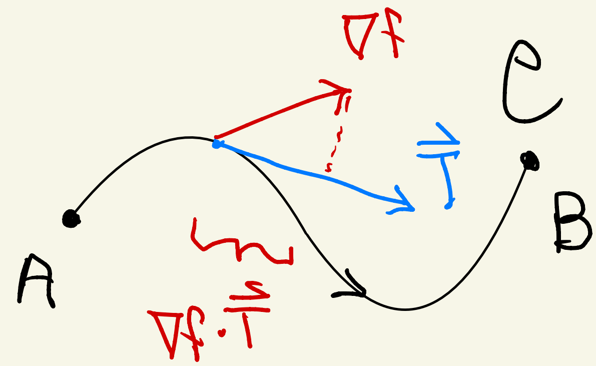
$$\frac{\partial f}{\partial x} = f_x = \partial_x f = \frac{\partial f}{\partial x}(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

We know (Math 21C) the Gradient takes a function $f : \mathbb{R}^3 \to \mathbb{R}$ (think of f as giving the temperature $f(\underset{\sim}{x})$ at $\underset{\sim}{x} = (x,y,z)$) and assigns to it the vector $\nabla f(x,y,z) = \overrightarrow{(M,N,P)}$ which points in direction of steepest increase of f.

- The first generalization of FTC involves the Gradient & we call it Conservation of Energy

(1) ∇: Conservation of Energy

$$\int_C \nabla f \cdot \vec{T}\, ds = f(B) - f(A)$$



Another way to say it:

We can evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$

if we can find an "anti-derivative" $f(\underset{\sim}{x})$ such that $\nabla f(\underset{\sim}{x}) = \vec{F}(\underset{\sim}{x}) = \overrightarrow{(M(\underset{\sim}{x}), N(\underset{\sim}{x}), P(\underset{\sim}{x}))}$ at every point $\underset{\sim}{x} = \overrightarrow{(x, y, z)}$. In this case

$$\boxed{\int_C \vec{F} \cdot \vec{T}\, ds = f(B) - f(A)}$$

Defn: We say a vector field $\vec{F} = \overrightarrow{(M, N, P)}$ is conservative if there exists f st $\nabla f = \vec{F}$

Important: Most Vector Fields $\vec{F}$ are NOT Conservative!

Questions to answer:

Q1: why is it true?

Q2: why is it called Conservation of Energy?

Q3: Given $\vec{F}$, how do you determine whether or not it is conservative?

Its easy to construct Examples:

Given $f(\underset{\sim}{x})$, just calculate $\nabla \vec{F} = \overrightarrow{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}$

Ex: Let $f(\underset{\sim}{x}) = xy^2z^3 \Rightarrow \nabla \vec{F} = \overrightarrow{(y^2z^3, 2xyz^3, 3xy^2z^2)}$

Let C: straight line from $(0,0,0)$ to $(1,1,1)$.

Show $\int_C \vec{F}\cdot\vec{T}\,ds = f(1,1,1) - f(0,0,0) = 1\cdot 1^2 \cdot 1^3 - 0 = 1$

Ic Parameterize: $x = t,\ y = t,\ z = t \quad 0 \le t \le 1$

$\vec{r}(t) = \overrightarrow{(t,t,t)},\ \vec{v}(t) = (1,1,1)$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^1 \vec{F}\cdot\vec{v}\,dt = \int_0^1 (y^2z^3, 2xyz^3, 3xy^2z^2)\cdot(1,1,1)\,dt$$

$$\int_0^1 t^5 + 2t^5 + 3t^5\,dt = \int_0^1 6t^5\,dt = 6\cdot\frac{t^6}{6}\Big]_0^1 = 1$$

✓

- Why is FTC-1 true?

Ans: Chain Rule

$$\frac{d}{dt} f(x(t),y(t),z(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \nabla f \cdot \overrightarrow{\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)}$$

$$= \nabla f \cdot \vec{v}(t)$$

Therefore –

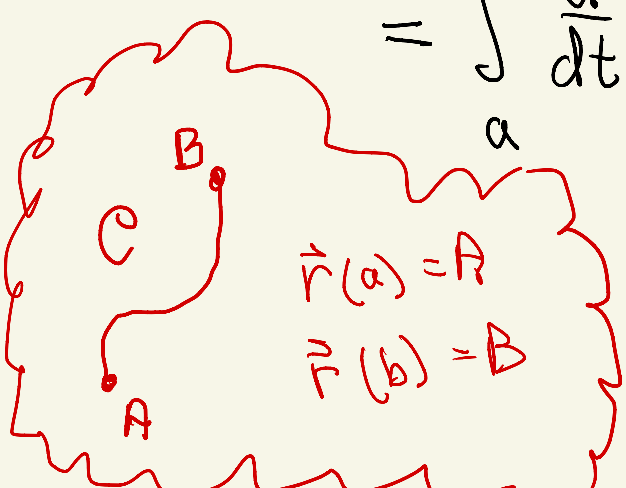
$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{v}\,dt = \int_a^b \underbrace{\nabla f\cdot\vec{v}}_{\frac{d}{dt}f(x(t),y(t),z(t))}\,dt$$

For C: $\vec{r}(t)$, $a \le t \le b$

assuming $\vec{F}$ conservative

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\underbrace{\vec{r}(a)}_{B}) - f(\underbrace{\vec{r}(b)}_{A})$$

Math 21B FTC

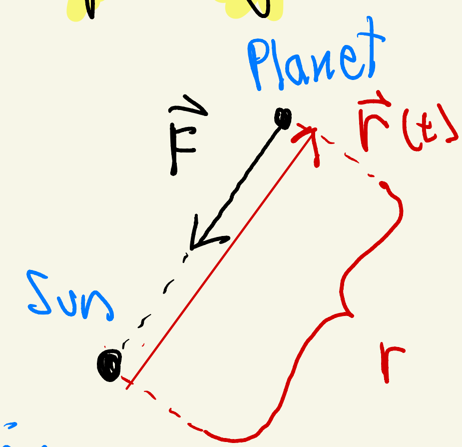


$$= f(B) - f(A)$$

Q2: Why is it called Conservation of Energy?

Example: Newton's Theory of Gravity

Recall: Newton explained the motion of the planets by assuming the sun was pulling with an inverse square force:



$$\vec{F} = M_p \vec{a} = \underbrace{-G \frac{M_p M_s}{r^2}}_{\text{magnitude}} \underbrace{\frac{\vec{r}}{r}}_{\text{direction}}$$

Claim: $\boxed{\nabla \frac{1}{|\vec{r}|} = -\frac{\vec{r}}{r^3}}$

$\vec{r} = \overrightarrow{(x, y, z)} \equiv \underset{\sim}{x}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2} = r$

Check: Let $f(x,y,z) = \frac{1}{|\vec{r}|} = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{|\underset{\sim}{x}|}$

It suffices to show $\frac{\partial f}{\partial x} = -\frac{x}{|\underset{\sim}{x}|^3}$

I.e. Then $\nabla f = -\overrightarrow{\left(\frac{x}{|\underset{\sim}{x}|^3}, \frac{y}{|\underset{\sim}{x}|^3}, \frac{z}{|\underset{\sim}{x}|^3}\right)} = -\frac{1}{|\underset{\sim}{x}|^3}\overrightarrow{(x, y, z)}$

$$= -\frac{\vec{r}}{r^3}$$

(Cont.) We prove $\nabla \frac{1}{|\underset{\sim}{x}|} = -\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|^3}$

Calculations involving partial derivatives of $r = |\underset{\sim}{x}|$ are done so often it is very convenient to have a quick way to do them...

For this – note $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

i.e. $\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2+y^2+z^2}}{\partial x} = \frac{1}{\cancel{2}} \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \cancel{2} x = \frac{x}{r}$ ✓

Thus: $\frac{\partial}{\partial x} \frac{1}{r} = \frac{\partial}{\partial x} r^{-1}(x,y,z) = -1 \frac{1}{r^2} \frac{\partial r}{\partial x} = -\frac{1}{r^2} \frac{x}{r} = -\frac{x}{r^3}$

Therefore: (symmetry) $\frac{\partial}{\partial x} \frac{1}{r} = -\frac{x}{r^3}$, $\frac{\partial}{\partial y} \frac{1}{r} = -\frac{y}{r^3}$, $\frac{\partial}{\partial z} \frac{1}{r} = -\frac{z}{r^3}$

$\Longrightarrow$ (implies) $\boxed{\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3}}$ ✓

Conclude:

$$\boxed{-G M_p M_s \frac{\vec{r}}{r^3} = G M_p M_s \nabla \frac{1}{r}}$$

Conclude: Newton's Gravitational Force Field is Conservative !

$$\vec{F} = -GM_sM_p \frac{\vec{r}}{r^3} = \nabla f = -\nabla P$$

$$f = GM_sM_p \frac{1}{r} \quad \left(f(x,y,z) = \frac{GM_sM_p}{\sqrt{x^2+y^2+z^2}} \right)$$

Now we can apply FTC-1

$$\underbrace{\int_{C} \vec{F}\cdot\vec{T}\,ds}_{\text{The work done by Gravitation Force as planet moves along } C} = \int_{C} \nabla f \cdot \vec{T}\,ds = \underbrace{f(B) - f(A)}_{\text{The change in } f \text{ = minus the change in "potential energy"}}$$

- In physics: P = Potential Energy $= -f$ associated with a conservative force $\vec{F} = \nabla f$ is defined to be $\boxed{P(\underset{\sim}{x}) = -f(\underset{\sim}{x})}$ ← minus sign

so $\boxed{\vec{F} = -\nabla P}$

Picture: The change in potential energy $\Delta P = P(B) - P(A)$ keeps track of (i.e., is exactly equal to) minus the work done by $\vec{F}$ along the motion.

Q: So why is the "work" defined by a line integral important in the first place?

Ans: If $\vec{F}$ is the only force acting on the planet, i.e., then motion $\vec{r}(t)$ satisfies

$$M_p \cdot \ddot{\vec{r}}(t) = \vec{F},$$

the work done is also equal to the change in kinetic energy —

Theorem: If $\vec{F} = -GM_sM_p \frac{\vec{r}}{r^3}$, and $M_s\ddot{\vec{r}} = \vec{F}$, then

$$\int_{e} \vec{F}\cdot\vec{T}\,ds = \underbrace{\frac{1}{2}M_pV_B^2 - \frac{1}{2}M_pV_A^2}_{\text{the change in Kinetic Energy}} = \Delta KE$$

Proof of Theorem: Assume $M_p \vec{a} = \vec{F}$ where $\vec{F} = -GM_sM_p \dfrac{\vec{r}}{r^3}$. Then

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{v}\,dt = \int_a^b M_p \frac{d\vec{v}}{dt}\cdot\vec{v}\,dt$$

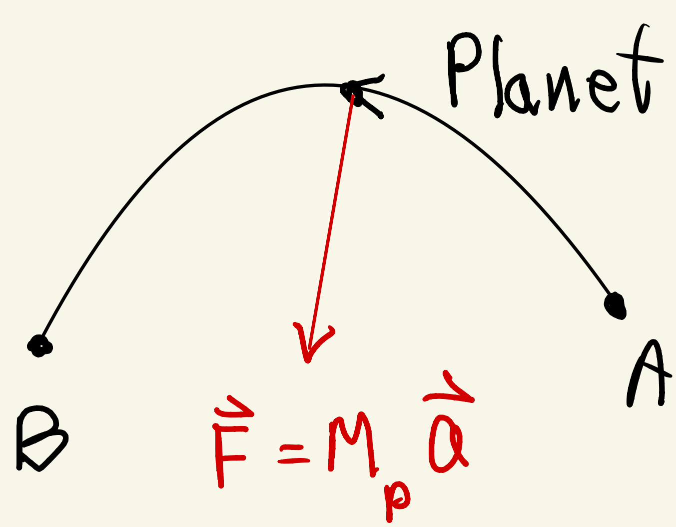
$$= M_p \int_a^b \frac{1}{2}\frac{d}{dt}\left(\vec{v}\cdot\vec{v}\right)dt = \frac{1}{2}M_p\,\vec{v}\cdot\vec{v}\Big]_{t=a}^{t=b}$$

$$= \frac{1}{2}M_p V_B^2 - \frac{1}{2}M_p V_A^2 = \Delta KE$$

$\vec{v}\cdot\vec{v} = v^2$

$v(a) = v_A$, $v(b) = v_B$

Change in Kinetic Energy



Conclude: If the planet moves according to

$$M_p \vec{a} = \vec{F}$$

where $\vec{F}$ is the gravitational force of the sun

$$\vec{F} = -GM_sM_p \frac{\vec{r}}{r^3},$$

then along the motion

$$\Delta KE = \tfrac{1}{2}M_pV_B^2 - \tfrac{1}{2}M_pV_A^2 = \int_e \vec{F}\cdot\vec{T}\,ds = -(P(B)-P(A)) = -\Delta PE$$

OR: $\Delta KE + \Delta PE = \Delta \text{Energy} = 0$

We say energy is conserved all along the motion. This is Conservation of Energy

Conclude: FTC-1 $\int_e \nabla f\cdot\vec{T}\,ds = f(B)-f(A)$ is the basis for the physical principle of Conservation of Energy —

Summary - Newton unified all the laws of planetary motion known in his lifetime — namely Kepler's three laws — by postulating an inverse square force law between neighboring masses.

- $F = M_p \vec{a}$ and $\vec{F} = -\frac{G M_p M_s}{r^2} \frac{\vec{r}}{r}$ led to "three miracles", Keplers three laws
- A "4th miracle" is that $\vec{F} = -\nabla P$

$$P(\underset{\sim}{x}) = -\frac{G M_p M_s}{r} \quad \text{potential energy}$$

So conservation of energy holds all along a planetary orbit $\vec{r}(t)$:

$$P(\vec{r}(t)) + \frac{1}{2} m \vec{v}(t)^2 = \text{constant}$$

This explains why orbits without a threshold energy remain trapped within the solar system!